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A Study of Fixed Point Theorem for Mappings Satisfying the Contractive Condition of Integral Type

By

Niraj Kumar
Research Scholar
P.G. Department of Mathematics
V.K.S. University, Ara

Prof. (Dr.) Alok Kumar
Professor and Head
P.G. Department of Mathematics
Maharaja College, Ara

INTRODUCTION

The study of fixed-point theorems for a map satisfying a contractive condition that did not require continuity at each point was initiated by Kannan [1] in 1968. This paper was a genesis for a multitude of fixed-point papers over the next two decades. Subsequently, several authors established fixed point theorems for a pair of maps.

In 1982, the notion of weakly commuting maps was initiated by Sessa [2]. It can be seen that commuting maps are weakly commuting but the converse is not true in general.

Then in 1986, Jungck [3] extended the concept of commutativity and weak commutativity by giving the notion of compatibility and showed that weakly commuting maps are compatible but the converse is not true in general. Many authors have obtained a lot of fixed-point theorems for compatible mappings satisfying contractive type conditions and assuming continuity of at least one of the mappings.

Definition 1.

Let X be a non-empty set and f and g be two self-mappings of X. A point $x \in X$ is called a coincidence point of f and g if and only if

$$fx = gx.$$

Definition 2.

Two self-mappings f and g on a metric space (X, d) are said to be weakly compatible if they commute at their coincidence points, that is,

$$f\,x = gx \text{ for some } x \in X \Rightarrow f\,gx = gfx.$$

Thus, two compatible maps are weakly compatible but the converse need not be true as shown in the following example.

Example

Let X = [0, 20] with the usual metric. Define the mappings $f, g : X \to X$ by

$$f(x) = \begin{cases} 0, & if\ x = 0, \\ x + 15, & if\ x \in [0,5] \\ x - 15, & if\ x \in (5,20] \end{cases}$$

$$g(x) = \begin{cases} 0, & if\ x \in \{0\} \cup (5,20] \\ 4, & if\ x \in (0,5] \end{cases}$$

Let $\{x_n\}$ be the sequence defined by

$$x_n = 5 + \frac{1}{n}, \ \forall n \in N,$$

Then

$$f(x_n) = x_n - 5 = \frac{1}{n}, \quad g(x_n) = 0$$

Therefore,

$$\lim_{n\to\infty} f(x_n) = 0 \ \text{ and } \ \lim_{n\to\infty} g(x_n) = 0$$

Also,

$$f(0) = 0 = g(0), \ \text{ and } \ fg(0) = gf(0),$$

Clearly, f and g are weakly compatible maps since they commute at their coincidence point x = 0.

On the other hand,

$$fg(x_n) = f(0) = 0 \ \text{ and } \ gf(x_n) = g(x_n - 5) = 4.$$

Consequently,

$$\lim_{n\to\infty} d(fg(x_n), gf(x_n)) = \lim_{n\to\infty} |fg(x_n) - gf(x_n)| = 4 \neq 0,$$

Hence, f and g are not compatible.

Example

Let (X, d) be metric space where

$$X = [0,2] \text{ and } d(x,y) = |x - y| \text{ for all } x, y \in X.$$

Define the mappings $A, B : X \to X$

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$$A(x) = \begin{cases} x, & \text{if } x \in [0, \frac{1}{3}), \\ \frac{1}{3}, & \text{if } x \in \left[\frac{1}{3}, 2\right], \end{cases}$$

$$B(x) = \frac{x}{x+1}, \text{ for all } x \in [0,2]$$

$$A(0) = B(0) = 0,$$

and hence

$$AB(0) = BA(0) = 0$$

Since A and B commute at their coincidence point x = 0, they are weakly compatible mappings.

Consider the sequence $\{x_n\}$ in X defined by

$$x_n = \frac{1}{2} + \frac{1}{n}, \quad n \geq 1,$$

$$\lim_{n\to\infty} A(x_n) = \frac{1}{3} \text{ and } \lim_{n\to\infty} B(x_n) = \frac{1}{3},$$

$$\lim_{n\to\infty} d(AB(x_n), BA(x_n)) = \left|\frac{1}{3} - \frac{1}{4}\right| \neq 0$$

Thus, A and B are not compatible.

Definition 3.

Two self-maps S and T of a set X are said to be occasionally weakly compatible if and only if there exists a point $x \in X$ which is a coincidence point of S and T at which S and T commute.

Clearly, weakly compatible maps are occasionally weakly compatible, but the converse is not always true.

Example

Let X = [0, ∞) with the usual metric. Define the mappings $S.T : X \to X$ by

$$S(x) = \begin{cases} 0, & x \in [0,1), \\ 2x, & x \in [1,\infty), \end{cases}$$

$$T(x) = \begin{cases} x, & x \in [0,1), \\ 1+x, & x \in [1,\infty), \end{cases}$$

Clearly, 0 and 1 are coincidence points since

$$S(0) = T(0) = 0 \quad \text{and} \quad S(1) = T(1),$$

But

$$ST(0) = TS(0) \text{ and } ST(1) = 4 \neq 3 = TS(1).$$

Hence, the pair {S,T} is occasionally weakly compatible but not weakly compatible.

Example

Let $\mathbb{R}$ be the usual metric space. Define the mappings U, V : $\mathbb{R} \to \mathbb{R}$ such that

$$U(x) = 2x \quad \text{and} \quad V(x) = x^2, \quad \text{for all } x \in \mathbb{R}.$$

Then

$$U(x) = V(x) \text{ for } x = 0, 2,$$

That is, 0 and 2 are coincidence points.

Here,

$$UV(0) = VU(0) = 0,$$

but

$$UV(2) \neq VU(2).$$

Hence, U and V are occasionally weakly compatible but not weakly compatible.

He we have some lemmas.

Lemmas

Lemma 1 (A1-Thagafi and Nasser, 2008).

Let X be a set and let f and g be occasionally weakly compatible self-maps of X. If f and g have a unique point of coincidence w = f(x) = g(x), then w is the unique common fixed point of f and g.

Lemma 2

Let A, B,S and T be self – mappings of a metric space (X, d). Let f be a summable, non-negative. Lebesgue integrable function from $\mathbb{R}^+$ into itself satisfying.

$$\int_0^S f(t)dt > 0, \ \forall s > 0$$

and $AX \subset TX$, $BX \subset SX$.

Assume that for a given ε > 0 there exists δ > 0 such that for all x, y in X,

$$\varepsilon < \int_0^{M(x,y)} f(t)dt < \varepsilon + \delta \Rightarrow \varepsilon \geq \int_0^{d(Ax,By)} f(t)dt \qquad (1)$$

and

$$\int_0^{M(x,y)} f(t)dt > 0 \Rightarrow \int_0^{d(Ax,By)} f(t)dt < \int_0^{M(x,y)} f(t)dt \qquad (2)$$

Where,

$$M(x,y) = \max\left\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \frac{d(Sx,By)+d(Ax,Ty)}{2}\right\} \quad (3)$$

Then for a fixed x_0 in X, the sequence $\{y_n\}$ in X defined by the rule

$$y_{2n} = Ax_{2n} = Tx_{2n+1}, y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \quad (4)$$

is a Cauchy sequence.

Theorem :

Let A, B, S, T, I, J be self – maps defined on a metric space (X, d) satisfying the following conditions :

$$AB(X) \subset J(X), \quad ST(X) \subset I(X), \quad (5)$$

For a given ε > 0, there exists δ > 0, such that for all x, y ∈x,

$$\varepsilon < \int_0^{M(x,y)} f(t)dt < \varepsilon + \delta \Rightarrow \int_0^{d(ABx.STy)} f(t)dt \le \varepsilon \quad (6)$$

and for $0 \le a \le \frac{1}{3}$,

$$\int_0^{d(ABx,STy)} f(t)dt < \alpha \int_0^{[d(Ix,Jy)+d(ABx,Ix)+d(STx,Jy)+d(Ix,STy)+d(ABx,Jy)]} f(t)dt \quad (7)$$

where f is a summable, non – negative, Lebesgue integrable function from $[0,\infty)$ into itself satisfying

$$\int_0^S f(t)dt > 0, \quad \forall s > 0 \quad (8)$$

and

$$M(x,y) = \max\left\{d(Ix,Jy), d(ABx,Ix), d(STy,Jy), \frac{d(Ix,STy)+d(ABx,Jy)}{2}\right\},$$

If one of the A(X), B(X), S(X), T(X), I(X) and J(X) is complete subspace of X, then

(i) AB and I have a coincidence point

or

(ii) ST and J have a coincidence point.

Further if the pairs {I, AB}and {J, ST] are occasionally weakly compatible, then

(iii) AB, ST, I and J have a unique common fixed point.

Proof :

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By virtue of (5) we define a sequence $\{Y_n\}$ in X as

$$y_{2n} = ABx_{2n} = Jx_{2n+1},\ y_{2n+1} = STx_{2n+1} = Ix_{2n+2}, \quad \forall n = 0,1,2,....$$

Suppose $d(y_{2n}, y_{2n+1}) = 0$, for some n, then $y_{2n} = y_{2n+1}$.

Hence

$$ABx_{2n} = Jx_{2n+1} = STx_{2n+1} = Ix_{2n+2}$$

and J and ST have a coincidence point.

Similarly, if $d(y_{2n+1}, y_{2n+2}) = 0,$, for some n, then

$$ABx_{2n+2} = Jx_{2n+3} = STx_{2n+1} = Ix_{2n+2}$$

and AB and I have a coincidence point.

Further assume that $d(y_n, y_{n+1}) \neq 0$ for each n. Then we have M(x, y) > 0. Otherwise, we have $d(y_n, y_{n+1}) = 0$, which is a contradiction. Hence using Lemma (2), $\{y_n\}$ is a Cauchy sequence in X.

Now suppose that J(X) is a complete subspace of X. Then the subsequence $\{y_{2n}\}$ contained in J(X) is convergent and has a limit in J(X), call it u and if we choose $\upsilon \in J^{-1}u$ then $J\nu = u$. Similarly, the subsequence $\{y_{2n+1}\}$ also converges to u.

Now we shall show that $ST\nu = u$. Let $d(ST\nu, u) > 0$. Then taking $x = x_{2n}$ and $y = \nu$ in (7), we get

$$\int_0^{d(ABx_{2n},ST_\nu)} f(t)dt < \alpha \int_0^{[d(Ix_{2n},J\nu)+d(ABx_{2n},Ix_{2n})+d(ST_\nu J\nu)+d(Ix_{2n},ST\nu)+d(u,ST\nu)+d(u,J\nu)]} f(t)dt$$

Using limit $n \to \infty$, we have

$$\int_0^{d(u,ST\nu)} f(t)dt < \alpha \int_0^{[d(u,J\nu)+d(u,u)+d(ST\nu,J\nu)+d(u,ST\nu)+d(u,J\nu)]} f(t)dt$$

$$= \alpha \int_0^{[d(u,u)+d(u,u)+d(ST\nu,u)+d(u,ST\nu)+d(u,u)]} f(t)dt$$

$$= \alpha \int_0^{[d(ST\nu,u)+d(u,ST\nu)]} f(t)dt = 2\alpha \int_0^{d(ST\nu,u)} f(t)dt$$

which is a contradiction since $0 \le \alpha \le \frac{1}{3}$. Hence, from (8), we get $ST\nu = u = J\nu$

Since $ST(X) \subseteq I(X)$ therefore, $ST\upsilon = u \Rightarrow u \in I(X).$ Hence, there exists some w in $I^{-1}u$ such that $Iw = u$.

Taking $x = w$ and $y = x_{2n+1}$ in (7), we have

$$\int_0^{d(ABw,STx_{2n+1})} f(t)dt < \alpha\int_0^{[d(Iw,Jx_{2n+1})+d(ABw,Iw)+d(STx_{2n+1},Jx_{2n+1})+d(Iw,Stx_{2n+1})+d(ABw,Jx_{2n+1})]} f(t)dt$$

Taking limit $n \to \infty$, we have,

$$\int_0^{d(ABw,u)} f(t)dt < \alpha\int_0^{[d(u,u)+d(ABw,u)+d(u,u)+d(u,u)+d(ABw,u)]} f(t)dt$$

$$= \alpha\int_0^{2d(ABw,u)} f(t)dt = 2\alpha\int_0^{d(ABw,u)} f(t)dt$$

This is a contradiction. Hence from (8).ABw = u = Iw.

Thus, the sets of coincidence points of the pairs (I, AB) and (J, ST)are non empty and u is unique. This proves (i) and (ii)

Now to prove (iii), not that (I, AB) and (J, ST) are occasionally weakly compatible and u = Jν = ST ν = ABw = Iw. Thus JSTν = ST Jν and ABIw = IABw.

If STu $\neq$ u then putting x = w and y = u in (7), we have

$$\int_0^{d(u,STv)} f(t)dt = \int_0^{d(ABw,STu)} f(t)dt$$

$$< \alpha\int_0^{d(Iw,Ju)+d(ABw,Iw)+d(STu,Ju)+d(Iw,STu)+d(ABw,Ju)} f(t)dt$$

$$= \alpha\int_0^{d(u,Ju)+d(u,u)+d(STu,Ju)+d(u,STu)+d(u,Ju)} f(t)dt$$

$$= 3\alpha\int_0^{d(u,STu)} f(t)dt$$

This is a contradiction for $\alpha \in \left[0, \frac{1}{3}\right]$. Thus from (8), we have STu = u = Ju.

Now, we will prove that ABu = STu and using condition (7), we have

$$\int_0^{d(ABu,STu)} f(t)dt < \alpha\int_0^{[d(Iu,Ju)+d(ABu,Ju)+d(STu,Ju)+d(Iu,STu)+d(ABu,Ju)]} f(t)dt$$

$$= 3\alpha\int_0^{d(ABu,STu)} f(t)dt$$

This is a contradiction and hence ABu = STu,

Therefore,

u = Ju = STu = ABu = Iu.

The uniqueness of the common fixed point u follows easily from condition (7). The same result holds if we assume that I(X) be complete instead of J(X).

Now if AB(X) is complete then by (5), $u \in AB(X) \subset J(X)$. Similarly, if ST(X) is complete then $u \in ST(X) \subset I(X)$. So, the theorem is established.

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